Mathematical Description of the Flows near the Bottom of the Ocean

A. Giniatoulline

Abstract—**We construct an explicit solution for a boundary value problem for a system of partial differential equations which describes small linearized motions of three-dimensional stratified flows in the half-space. For large values of t, we obtain uniform asymptotical decompositions of the solutions on an arbitrary compact in the half-space. In the vicinity of the boundary plane, we establish the asymptotical properties of the boundary layer type: we can observe a worsening of the decay in the approximation to the bottom. The results can be used in the meteorological modelling of water flows near the bottom of the Ocean, as well as Atmosphere flows near the Earth surface.**

Index Terms—**Atmospheric modelling and numerical prediction, boundary layer, geophysics, partial differential equations, physical oceanography, stratified fluid.**

I. INTRODUCTION

We consider a system of equations of the form

$$\left|\begin{array}{l} \rho_* \dfrac{\partial v_1}{\partial t} + \dfrac{\partial p}{\partial x_1} = 0 \\[2mm] \rho_* \dfrac{\partial v_2}{\partial t} + \dfrac{\partial p}{\partial x_2} = 0 \\[2mm] \rho_* \dfrac{\partial v_3}{\partial t} + g\rho + \dfrac{\partial p}{\partial x_3} = 0 \\[2mm] \dfrac{\partial \rho}{\partial t} - \dfrac{N^2 \rho_*}{g} v_3 = 0 \\[2mm] \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0 \end{array}\right. \qquad (1)$$

where $v(x,t)$ is a velocity field of a fluid (Ocean or Atmosphere) with components $v_1(x,t)$, $v_2(x,t)$, $v_3(x,t)$; $p(x,t)$ is the scalar field of the dynamic pressure; $\rho(x,t)$ is the dynamic density and ρ_*, g, N are positive constants. The equations (1) are deduced in [1], [2] under the assumption that the function of stationary distribution of density is performed by the exponentially decreasing function $\rho_* e^{-Nx_3}$, which corresponds to the Boltzmann distribution. The system (1) can be considered as modelling

the linearized movement of three-dimensional fluid in a homogeneous gravitational field (Ocean or Atmosphere), as it can be visualized in the following Fig. 1:
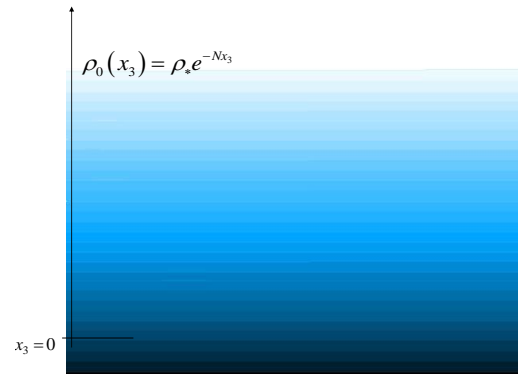

Fig. 1. Initial distribution of density as a function of x_3.

Some physical properties of the flows described by (1) can be found, for example, in [3]-[5].

The Cauchy problem for (1) was considered in [6], where the explicit solution was constructed and L_p-estimates were obtained. The half-space domain for (1) was considered in [7], where the solution was constructed using unilateral Fourier transform; the uniqueness of the solutions was proved in a class of growing functions, and it was established that the solution decays for large t as $1/\sqrt{t}$. The spectral properties of the differential operator of (1) for compressible viscous case were studied in [8]. In this work, we consider the system (1) in the half-space

$$R_+^3 = \left\{ (x_1, x_2, x_3) : (x_1, x_2) \in R^2, x_3 \geq 0 \right\}$$

together with the initial conditions

$$\begin{cases} \vec{v}\big|_{t=0} = \vec{v}^{\,0}(x), \ \operatorname{div} \vec{v}^{\,0} = 0 \\ \rho\big|_{t=0} = 0 \end{cases} \qquad (2)$$

and the boundary condition

$$p\big|_{x_3=0} = 0, \ t \geq 0 \qquad (3)$$

Without loss of generality, we may assume $\rho_* = g = N = 1$. This can be achieved by the following change of scale, where we modify the unknown functions of

velocity and density and use the same notation for modified functions: $\vec{v} := \rho_* \vec{v}$, $\rho := g\rho$.

In that way, instead of system (1), we will consider the system:

$$\begin{cases} \dfrac{\partial v_1}{\partial t} + \dfrac{\partial p}{\partial x_1} = 0 \\[2mm] \dfrac{\partial v_2}{\partial t} + \dfrac{\partial p}{\partial x_2} = 0 \\[2mm] \dfrac{\partial v_3}{\partial t} + \rho + \dfrac{\partial p}{\partial x_3} = 0 \\[2mm] \dfrac{\partial \rho}{\partial t} - v_3 = 0 \\[2mm] \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0 \end{cases} \qquad (4)$$

II. Construction of the Solution

To construct the solution of (4) in the half-space R_+^3, we will solve an ordinary differential equation with respect to x_3. Here we proceed in a different way if we compare it with the solution from [7], where "unilateral" sine and cosine Fourier transforms were used with respect to x_3.

Supposing that the initial data $\vec{v}^0(x)$ are smooth and sufficiently well decreasing functions for $|x| \to \infty$, we apply Fourier transform $(x_1, x_2) \to (\xi_1, \xi_2)$ and Laplace transform $t \to \lambda$ for (4). In this way, we obtain the following system:

$$\begin{cases} \lambda \tilde{v}_1 - \hat{v}_1^0 + i\xi_1 \tilde{P} = 0 \\[2mm] \lambda \tilde{v}_2 - \hat{v}_2^0 + i\xi_2 \tilde{P} = 0 \\[2mm] \lambda \tilde{v}_3 - \hat{v}_3^0 + \tilde{\rho} + \dfrac{\partial \tilde{P}}{\partial x_3} = 0 \\[2mm] \lambda \tilde{\rho} - \tilde{v}_3 = 0 \\[2mm] i\xi_1 \tilde{v}_1 + i\xi_2 \tilde{v}_2 + \dfrac{\partial \tilde{v}_3}{\partial x_3} = 0 \end{cases} \qquad (5)$$

Let us denote

$$\tilde{P} = \tilde{P}(\xi', x_3, \lambda) , \ \ \xi' = (\xi_1, \xi_2) , \ \ |\xi'| = \sqrt{\xi_1^2 + \xi_2^2} .$$

In particular, for the function $\tilde{P}$ from (5) we have the following ordinary differential equation:

$$\frac{\partial^2 \tilde{P}}{\partial x_3^2} - \left(\frac{\lambda^2 + 1}{\lambda^2} \right) |\xi'|^2 \, \tilde{P} = -\frac{1}{\lambda^2} \frac{\partial \hat{v}_3^0}{\partial x_3} . \qquad (6)$$

In terms of Fourier-Laplace transform, the boundary condition (3) will take the form

$$\tilde{P}\Big|_{x_3=0} = 0 . \qquad (7)$$

We denote the characteristic roots of the equation (6) as

$$\pm a , \ \ a = \frac{\sqrt{\lambda^2 + 1}}{\lambda} |\xi'| .$$

In what follows, we will consider $\operatorname{Re}\lambda \geq \lambda_0 > 0$.

The general solution of (6) contains two arbitrary constants which depend on λ and ξ' .

One of them can be defined from the condition (7), and the other - from the condition $\lim\limits_{x_3 \to \infty} \tilde{P} = 0$. In this way, we obtain

$$\tilde{P}(\xi', x_3, \lambda) = \frac{-1}{2|\xi'|\lambda\sqrt{\lambda^2 + 1}} \cdot \\ \cdot \int_0^\infty \left(e^{-a(\eta + x_3)} - e^{-a|\eta - x_3|} \right) \frac{\partial \hat{v}_3^0}{\partial \eta}(\xi', \eta) d\eta \qquad (8)$$

If we substitute the representation (8) in (5), we obtain a linear algebraic system for the functions $\tilde{v}_j(\xi', x_3, \lambda)$, $j = 1, 2, 3$, $\tilde{\rho}(\xi', x_3, \lambda)$. After solving that system, we have the following:

$$\tilde{v}_j(\xi', x_3, \lambda) = \frac{\hat{v}_j^0}{\lambda} + \frac{1}{2|\xi'|\lambda\sqrt{\lambda^2 + 1}} \cdot \\ \cdot \int_0^\infty \left(e^{-a(\eta + x_3)} - e^{-a|\eta - x_3|} \right) \frac{i\xi_j}{\lambda} \frac{\partial \hat{v}_3^0}{\partial \eta}(\xi', \eta) d\eta , j = 1, 2,$$

$$\tilde{v}_3(\xi', x_3, \lambda) = \frac{\lambda \hat{v}_3^0}{\lambda^2 + 1} + \frac{1}{2\lambda(\lambda^2 + 1)} \cdot \\ \cdot \int_0^\infty \left(e^{-a(\eta + x_3)} + \operatorname{Sign}(\eta - x_3) e^{-a|\eta - x_3|} \right) \frac{\partial \hat{v}_3^0}{\partial \eta}(\xi', \eta) d\eta ,$$

$$\tilde{\rho}(\xi', x_3, \lambda) = \tilde{v}_3(\xi', x_3, \lambda) \big/ \lambda$$

It is easy to see that, for the inverse Fourier transform $\xi' \to x'$, the identities hold:

$$F^{-1}\left[\frac{1}{|\xi'|} e^{-\frac{\sqrt{\lambda^2 + 1}}{\lambda}|\xi'|x_3} \right] = \frac{1}{|x|} \frac{\lambda}{\sqrt{\lambda^2 + \dfrac{x_3^2}{|x|^2}}} ,$$

$$F^{-1}\left[e^{-\frac{\sqrt{\lambda^2 + 1}}{\lambda}|\xi'|x_3} \right] = \frac{x_3}{|x|^3} \frac{\lambda^2 \sqrt{\lambda^2 + 1}}{\sqrt{\left(\lambda^2 + \dfrac{x_3^2}{|x|^2} \right)^3}} .$$

In this way, the inverse Fourier-Laplace

transform $\begin{pmatrix} \xi' \to x' \\ \lambda \to t \end{pmatrix}$ will take the form:

$$L^{-1}\left[F^{-1}\left[\frac{1}{|\xi'|\lambda\sqrt{\lambda^2+1}} e^{-\frac{\sqrt{\lambda^2+1}}{\lambda}|\xi'|x_3} \right] \right] = \frac{1}{|x|}Q(k,t),$$

$$k = \frac{|x_3|}{|x|}, Q(k,t) = J_0(t)*J_0(kt) = \int_0^t J_0(t-\tau)J_0(k\tau)d\tau;$$

$$L^{-1}\left[F^{-1}\left[\frac{1}{\lambda(\lambda^2+1)} e^{-\frac{\sqrt{\lambda^2+1}}{\lambda}|\xi'|x_3} \right] \right] = \frac{1}{|x|^2}R(k,t),$$

$$R(k,t) = -J_0(t)*J_1(kt) = -\int_0^t J_0(t-\tau)J_1(k\tau)d\tau;$$

$$(9)$$

where J_0 and J_1 are Bessel functions.

We use (9) and apply the inverse Fourier and Laplace transforms to (8). Thus, for the function $P(x,t)$ we obtain the following:

$$P(x,t) = \frac{-1}{4\pi}\int_{R^2} dy' \cdot \int_0^\infty \begin{bmatrix} \frac{1}{r_2}Q\left(\frac{|x_3+\eta|}{r_2},t\right) - \\ -\frac{1}{r_1}Q\left(\frac{|x_3-\eta|}{r_1},t\right) \end{bmatrix} \frac{\partial v_3^0}{\partial\eta} d\eta,$$

$$r_1 = \sqrt{|x'-y'|^2 + (x_3-\eta)^2},$$

$$r_2 = \sqrt{|x'-y'|^2 + (x_3+\eta)^2}. \qquad (10)$$

Analogously, for the rest of the components of the solution for the problem (2)-(4), we obtain the representations:

$$v_j(x,t) = v_j^0(x) + \frac{1}{4\pi}\int_{R^2} dy'\int_0^\infty Q_0(x,y',\eta,t)\cdot$$

$$\cdot\frac{\partial^2 v_3^0(y',\eta)}{\partial\eta\partial y_j} d\eta, \quad j=1,2,$$

$$Q_0(x,y',\eta,t) = \int_0^t \begin{bmatrix} \frac{1}{r_2}Q\left(\frac{|x_3+\eta|}{r_2},\tau\right) - \\ -\frac{1}{r_1}Q\left(\frac{|x_3-\eta|}{r_1},\tau\right) \end{bmatrix} d\tau,$$

$$v_3(x,t) = v_3^0(x)\cos t +$$

$$+\frac{1}{4\pi}\int_{R^2} dy'\int_0^\infty \begin{bmatrix} \frac{1}{r_2^2}R\left(\frac{|x_3+\eta|}{r_2},t\right) - \\ -\frac{1}{r_1^2}R\left(\frac{|x_3-\eta|}{r_1},t\right) \end{bmatrix} \frac{\partial v_3^0(y',\eta)}{\partial\eta} d\eta,$$

$$\rho(x,t) = \int_0^t v_3(x,\tau)d\tau. \qquad (11)$$

III. ASYMPTOTICAL DECOMPOSITION ON A COMPACT SET FOR LARGE VALUES OF TIME AND BOUNDARY LAYER PROPERTIES AT THE BOTTOM

It is easy to see that the obtained representations (10)-(11), essentially, have the same qualitative form. Thus, without loss of generality, we will investigate in detail the asymptotical behavior of one of the components of the solution. In particular, let us consider the function $P(x,t)$.

We introduce the function

$$u(x) = \frac{1}{2}\left\{ \frac{\partial v_3^0}{\partial x_3}(x',x_3) + \frac{\partial v_3^0}{\partial x_3}(x',-x_3) \right\}.$$

In this way, we can represent the function $P(x,t)$ from (10), as follows:

$$P(x,t) = P_1(x,t) + P_2(x,t), \qquad (12)$$

where

$$P_1(x,t) = \frac{1}{4\pi}\int_{R^3} \frac{1}{|y|}Q\left(\frac{|y_3|}{|y|},t\right)\frac{\partial v_3^0(x-y)}{\partial x_3}dy,$$

$$P_2(x,t) = \frac{1}{2\pi}\int_{R^2} dy'\int_{x_3}^\infty u(x'-y',y_3-x_3)\frac{1}{|y|}Q\left(\frac{|y_3|}{|y|},t\right)dy_3.$$

We observe that $P_1(x,t)$ coincides with the representation of the solution $P(x,t)$ for the Cauchy problem from [6] for the case when the initial data $\vec{v}^0(x)$ are extended till the whole space R^3. Such extension preserves the property of the solenoidality (see [9]).

Now, let us consider the function $P_2(x,t)$. In other terms, we consider the function

$$F(x,t) = \frac{1}{2\pi}\int_{R^2} dy'\int_{x_3}^\infty f(x'-y',y_3-x_3)\cdot$$

$$\cdot\frac{1}{|y|}Q\left(\frac{|y_3|}{|y|},t\right)dy_3,$$

where $\qquad (13)$

$$\frac{1}{|y|}Q\left(\frac{|y_3|}{|y|},t\right) = \frac{1}{|y|}J_0(t)*J_0\left(\frac{|y_3|}{|y|}t\right) =$$

$$= \frac{2}{\pi}\int_{\frac{|y_3|}{|y|}}^\infty \frac{\sin(t\mu)}{\sqrt{1-\mu^2}\sqrt{\mu^2-\frac{|y_3|^2}{|y|^2}}}d\mu. \quad \text{(See [6], [7])}$$

After passing in (13) to cylindrical coordinates, we obtain

$$F(x,t) = \frac{1}{2\pi}\cdot$$

$$\cdot\int_{x_3}^\infty dy_3 \begin{bmatrix} \int_0^\infty \frac{\rho}{r}Q\left(\frac{|y_3|}{r},t\right)d\rho \cdot \\ \cdot\int_0^{2\pi} f(x_1-\rho\cos\varphi, x_2-\rho\sin\varphi, y_3-x_3) \end{bmatrix} d\varphi, \qquad (14)$$

Now, let us suppose that the function $f(x)$ satisfies the condition

$$\int_{R_+^3}(1+|x|)^{3l+2}|D_x^\beta f(x)|dx \le C_\beta^* \quad, \quad 0\le|\beta|\le l+1 \qquad (15)$$

for some constants $C_\beta^* > 0$.

We introduce the notation

$$R_h^3 = \left\{x=(x',x_3): x'\in R^2 \,, \, x_3\ge h\right\}$$

and observe that the internal integral with respect to φ in (14) is an even function for ρ .

We integrate by parts $l+1$ times in (14) with respect to y_3 and obtain thus the estimate

$$|F(x,t)|\le \frac{b}{(h^2t)^{l+1}} \qquad (16)$$

for all $t\ge t_0 >0$, $x\in K\subset R_h^3$, being K a compact.

If $\vec{v}^0(x)\in C_0^\infty(R_+^3)$ then, from [6], [7] we have the uniform estimate for all $x\in R_+^3$:

$$|P_1(x,t)|\le \frac{C}{\sqrt{t}} \quad. \qquad (17)$$

To obtain the estimate (17), the expression of the main term of the asymptotic expansion was used ([10]):

$$\int_\lambda^1 \frac{\sin(t\mu)}{\sqrt{1-\mu^2}\sqrt{\mu^2-\lambda^2}}d\mu =$$

$$= C(\lambda)\frac{\sin\left(t\lambda-\dfrac{\pi}{4}\right)}{\sqrt{t}}+O(t^{-1}) \,, \, t\to\infty$$

Now, instead of studying just the main term of the asymptotic decomposition, we can consider $2l$ first terms.

In this way, we obtain the following estimate for $t\to\infty$, $\forall x\in K\subset R_+^3$

$$P_1(x,t)=\sum_{k=1}^{2l}\left(B_k^1(x)\cos t + B_k^2(x)\sin t\right)\cdot t^{-\frac{k}{2}} + \\ +O(t^{-l-1}) \qquad (18)$$

We note that, for the rest of the components of the solution (11) the procedure is analogous.

Therefore, summing up (12), (16) and (18), we have proved the following result:

Theorem.

Let $\vec{v}^0(x)\in C^{2l+4}(R_+^3)$ for some natural fixed l. Additionally, suppose that there exist positive constants C_β such that the condition holds:

$$\int_{R_+^3}(1+|x|)^{3l+2}|D_x^\beta \vec{v}^0(x)|dx \le C_\beta \quad, \quad 0\le|\beta|\le 2l+4 \,.$$

Let K be a compact such that

$$K\subset R_h^3 \,, \, R_h^3 = \left\{x=(x',x_3): x'\in R^2 \,, \, x_3\ge h\right\}\,.$$

Then, for all $h>0$, $K\subset R_h^3$, the asymptotic decomposition for the solution of the problem (2), (3), (4) has the following form for $t\to\infty$:

$$\vec{v}(x,t)=\sum_{k=1}^{2l}\left(\vec{A}_k^1(x)\cos t + \vec{A}_k^2(x)\sin t\right)\cdot t^{-\frac{k}{2}} + \\ +\vec{A}_0(x,t)+O(t^{-l-1})$$

$$P(x,t)=\sum_{k=1}^{2l}\left(B_k^1(x)\cos t + B_k^2(x)\sin t\right)\cdot t^{-\frac{k}{2}} + \\ +B_0(x,t)+O(t^{-l-1})$$

where $\vec{A}_k^i$, B_k^i are continuous functions on the compact K, and there exist the positive constants a_0 , b_0 such that for all $t\ge t_0 >0$ the estimate is valid:

$$|\vec{A}_0(x,t)|\le \frac{a_0}{(h^2t)^{l+1}} \,, \quad |B_0(x,t)|\le \frac{b_0}{(h^2t)^{l+1}}\,.$$

IV. CONCLUSION

We have studied the asymptotical behavior of the solutions of boundary value problem of the type (3) in the half-space. In other terms, we have studied the vanishing velocity of the perturbation defined by the initial data (2) for the case of the existence of an obstacle (bottom of the Ocean or the Earth surface for the case of the Atmosphere). In this process, there occurs a boundary layer-type effect in the asymptotical decompositions of the solutions for large values of t with respect to the small parameter $1/t$ near the plane $x_3=0$.

CONFLICT OF INTEREST

This work was carried out without any conflict of interest.

AUTHOR CONTRIBUTIONS

As a single-author article, the research of this work, the writing of the paper and also the approval of the final version was conducted by A. Giniatoulline.

ACKNOWLEDGMENT

REFERENCES

[1] D. Tritton, *Physical Fluid Dynamics*, Oxford: Oxford UP, 1990.
[2] P. Kundu, *Fluid Mechanics*, New York: Acad. Press, 1990.
[3] V. Birman and E. Meiburg, "On gravity currents in stratified ambient," *Phys.Fluids*, vol. 19, pp. 602–612, 2007.
[4] T. Dauxois and W. Young, "Near-critical reflection of internal waves," *J.Fluid Mech.*, vol. 90, pp. 271–295, 1999.
[5] J. Flo, M. Ungari, and J. Bush, "Spin-up from rest in a stratified fluid: Boundary flows," *J.Fluid Mech.* vol. 472, pp. 51–82, 2002.
[6] A. Giniatoulline and O. Zapata, "On some qualitative properties of stratified flows," *Contemporary Mathematics*, vol. 434, pp. 193-204, 2007.
[7] A. Giniatoulline and E. Mayorga, "On some properties of the solutions of the problem modelling stratified ocean and atmosphere flows in the half-space," *Revista Colombiana Mat.*, vol. 41, no. 2, pp. 333–344, 2007.
[8] A. Giniatoulline and T. Castro, "On the spectrum of the operator of inner waves in a viscous compressible stratified fluid", *J. Math. Sci. Univ. Tokyo*, vol. 19, pp. 313–323, 2012.
[9] V. Solonnikov, "Estimates for solutions of a non-stationary system of navier-stokes equations," *Trud Mat. Inst. Steklov*, vol. 70, pp. 213-317, 1964.
[10] E. Copson, *Asymptotic Expansions*, Cambridge: CUP, 2004.

Andrei Giniatoulline received his undergraduate, MSc, and PhD degrees from Friendship University in Moscow, Russia. Since 1993, he has taught undergraduate and graduate level courses at the Department of Mathematics of Los Andes University in Bogota, Colombia, where he holds the position of Associate Professor. His research interests are in the areas of mathematical physics, with an emphasis on hydrodynamics and applied functional analysis, mathematical description of the environmental problems, particularly, the flows in the atmosphere and the ocean. He has delivered invited lectures on the subject of flow dynamics and spectral theory at universities in Bolivia, Brazil, Colombia, Japan, Malaysia, Russia, Spain, South Korea, Turkey and other countries.